



NTNU

Norwegian University of
Science and Technology

RANDOMNESS 3

TTM4205 – Lecture 4

Tjerand Silde

31.08.2023

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Reference Group

I am looking for (at least) three students to form a reference group in this course, preferably students from different programs. We will meet three times during the semester, and your feedback is extremely valuable.

Send me an email and/or talk to me in the break :)

Open PhD Position

The image is a promotional graphic for a PhD position. It features the NTNU logo (a blue square with a white circle) and the text 'NTNU' in a large, bold, black serif font. Below this, 'Norwegian University of Science and Technology' is written in a smaller, black sans-serif font. The background is a light, faded image of a building with a tall spire and a flag. The text 'The Department of Information Security and Communication Technology (IHK) has a vacancy for a' is centered in a black sans-serif font. Below that, 'PhD Candidate in Cryptography Engineering' is written in a large, bold, black sans-serif font.

NTNU
Norwegian University of
Science and Technology

The Department of Information Security and Communication Technology (IHK) has a
vacancy for a

PhD Candidate in Cryptography Engineering

Figure: <https://www.jobbnorge.no/en/available-jobs/job/246480/phd-candidate-in-cryptography-engineering>

Uniped Observation

I am completing a course in University Pedagogy (Uniped) this year, and next week, on Tuesday, September 5th, I have so-called *collegial coaching*. This means that a few other lecturers from different departments at NTNU will be observing my lecture and will provide feedback to me afterward. They are **not** observing you.

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How do we check if a number is prime?

Deterministic Methods

- ▶ Brute Force
- ▶ Sieving methods
- ▶ Wilson's Theorem?

Brute Force Testing

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This is infeasible to compute! 2^{128} is considered impossible.

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First, keep 2 and remove all even numbers. Then, keep 3 and remove all multiples of three. 4 is already removed. Keep 5 and remove all multiples of five. 6 has already been removed. ...

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It still requires exponential work to check all possibilities!

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But it is possible to use similar techniques to speed it up.

Randomized Methods

- ▶ Monte Carlo algorithms
- ▶ The Miller-Rabin method

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Some commonly used algorithms: Soloway-Strassen, Fermat (**warning**: Carmichael numbers) and Miller-Rabin.

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If we sample λ random values a , the Miller-Rabin primality testing algorithm has $\frac{1}{4}^\lambda$ chance of being wrong every time.

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4. If all checks succeeds, then output: *probably prime*.

Primality Testing Failures

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Sometimes it is a mix between fixed a 's and freshly sampled a 's, still giving the adversary a good chance to fool the test.

Primality Testing in OpenSSL

Prime and Prejudice: Primality Testing Under Adversarial Conditions

Martin R. Albrecht¹, Jake Massimo¹, Kenneth G. Paterson¹, and Juraj Somorovsky²

¹ Royal Holloway, University of London

² Ruhr University Bochum, Germany

`martin.albrecht@rhul.ac.uk`, `jake.massimo.2015@rhul.ac.uk`, `kenny.paterson@rhul.ac.uk`,
`juraj.somorovsky@rub.de`

Figure: <https://eprint.iacr.org/2018/749.pdf>

The Need for Secure Primality Testing

Safety in Numbers: On the Need for Robust Diffie-Hellman Parameter Validation

Steven Galbraith¹, Jake Massimo², and Kenneth G. Paterson²

¹ University of Auckland

² Royal Holloway, University of London

s.galbraith@auckland.ac.nz, jake.massimo.2015@rhul.ac.uk,

kenny.paterson@rhul.ac.uk

Figure: <https://eprint.iacr.org/2019/032.pdf>

Secure Primality Testing API

A Performant, Misuse-Resistant API for Primality Testing

Jake Massimo¹ and Kenneth G. Paterson²

¹ Information Security Group,
Royal Holloway, University of London
`jake.massimo.2015@rhul.ac.uk`

² Department of Computer Science,
ETH Zurich
`kenny.paterson@inf.ethz.ch`

Figure: <https://eprint.iacr.org/2020/065.pdf>

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How do we factor
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Deterministic Methods

Some trivial ways to attack an RSA moduli n :

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Randomness comes to the rescue in this situation as well!

Randomized Methods

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Then we *might* find a factor of n by computing the greatest common divisor between n and $a - b$ and $a + b$.

Number Field Sieve

The running time of the Number Field Sieve is

$$\exp\left(\left(\frac{64}{9}\right)^{1/3}(\log n)^{1/3}(\log \log n)^{2/3}(1 + o(1))\right)$$

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Factoring as a service: In 2015, it was possible to factor 512 bit RSA keys in less than four hours.

Factoring as a Service

Factoring as a Service

Luke Valenta, Shaanan Cohney, Alex Liao,
Joshua Fried, Satya Bodduluri, Nadia Heninger

University of Pennsylvania

Figure: <https://eprint.iacr.org/2015/1000.pdf>

State of the Art

The state of the art in integer factoring and breaking public key cryptography

Fabrice Boudot¹, Pierrick Gaudry², Aurore Guillevic², Nadia Heninger³, Emmanuel Thomé²,
and Paul Zimmermann²

¹Université de Limoges, XLIM, UMR 7252, F-87000 Limoges, France

²Université de Lorraine, CNRS, Inria, LORIA, F-54000 Nancy, France

³University of California, San Diego, USA

Figure: <https://hal.science/hal-03691141/document>

RSA Failures in Practice

How do we break the following RSA keys?

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- ▶ Same seed when sampling primes

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RSA Failures in Practice

How do we break the following RSA keys?

- ▶ Same seed when sampling primes
- ▶ Same seed + added entropy between sampling
- ▶ Low entropy RNG or PRNG from known algorithm
- ▶ Related primes from known algorithm

RSA, DH and DSA in the Wild*

Nadia Heninger

University of California, San Diego, USA

Figure: <https://eprint.iacr.org/2022/048.pdf>

Fermat Factorization in the Wild

Hanno Böck

January 8, 2023

Figure: <https://eprint.iacr.org/2023/026.pdf>

Shared Prime Factors

Ron was wrong, Whit is right

Arjen K. Lenstra¹, James P. Hughes²,
Maxime Augier¹, Joppe W. Bos¹, Thorsten Kleinjung¹, and Christophe Wachter¹

¹ EPFL IC LACAL, Station 14, CH-1015 Lausanne, Switzerland

² Self, Palo Alto, CA, USA

Figure: <https://eprint.iacr.org/2012/064.pdf>

Shared Prime Factors

Mining Your Ps and Qs: Detection of Widespread Weak Keys in Network Devices

Nadia Heninger^{†*} Zakir Durumeric^{‡*} Eric Wustrow[‡] J. Alex Halderman[‡]
[†] *University of California, San Diego*
nadiah@cs.ucsd.edu [‡] *The University of Michigan*
{zakir, ewust, jhalderm}@umich.edu

Figure: Check out the blog post, paper and slides: 1) <https://freedom-to-tinker.com/2012/02/15/new-research-theres-no-need-panic-over-factorable-keys-just-mind-your-ps-and-qs>, 2) <https://factorable.net/weakkeys12.extended.pdf>, 3) <https://crypto.stanford.edu/RealWorldCrypto/slides/nadia.pdf>

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De-Randomized Crypto

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We DO need randomness for key generation. However:

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- ▶ Counters + master seed + hashing
- ▶ HMAC with key for deterministic MAC
- ▶ Hedging techniques (next slide)

Shared Prime Factors

Hedged Public-Key Encryption: How to Protect Against Bad Randomness

Mihir Bellare* Zvika Brakerski† Moni Naor‡ Thomas Ristenpart§
Gil Segev¶ Hovav Shacham|| Scott Yilek**

April 21, 2012

Figure: <https://www.cs.utexas.edu/~hovav/dist/hedge.pdf>

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Random Number Generation

Check the quality of the built-in RNG that you rely on:

- ▶ How does it collect randomness?
- ▶ Is the RNG seeded / pre-seeded?
- ▶ How much entropy does it provide?
- ▶ Does it warn you about issues?
- ▶ Is it cryptographically secure?
- ▶ (Linux's */dev/random* vs */dev/urandom*)

Faulty Voting Randomness

A faulty PRNG in a voting system

– a real-world cryptographic disaster

Kristian Gjøsteen

Department of Mathematical Sciences

Norwegian University of Science and Technology

Real World Crypto, January 2018

Figure: https://youtu.be/xq_6ey2JGAE?feature=shared

Pseudo-Random Number Generation

Check the quality of the built-in PRNG that you rely on:

- ▶ Does it rely on a proper RNG as seed? Is it pre-seeded?
- ▶ Is the PRNG cryptographically secure? NIST-approved?
- ▶ Verify the output: Do values repeat? Correct bit-size?
- ▶ Which library/version is used? Known vulnerabilities?

Some good resources are available at <https://github.com/veorq/cryptocoding#use-strong-randomness>.



**National Institute of
Standards and Technology**

Technology Administration
U.S. Department of Commerce

**Special Publication 800-22
Revision 1a**

A Statistical Test Suite for Random and Pseudorandom Number Generators for Cryptographic Applications

Figure: <https://csrc.nist.gov/pubs/sp/800/22/r1/upd1/final>

Choice of Primitives

Check the cryptographic primitive that you rely on:

- ▶ Does it rely on a proper PRNG? Is it pre-seeded?
- ▶ Is it the newest/most secure primitive? NIST-approved?
- ▶ Verify the output: Do values repeat? Correct bit-size?
- ▶ Which library/version is used? Known vulnerabilities?
- ▶ Are there de-randomized algorithms available instead?

Rolling Your Own Crypto

Security Cryptography Whatever

The Great "Roll Your Own Crypto" Debate with Filippo Valsorda

JULY 31, 2021 SECURITY, CRYPTOGRAPHY, WHATEVER



Figure: <https://securitycryptographywhatever.buzzsprout.com/1822302/8953842-the-great-roll-your-own-crypto-debate-with-filippo-valsorda>

Questions?