Norwegian University of Science and Technology

RANDOMNESS 1

TTM4205 – Lecture 2

Tjerand Silde

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Randomness

Randomness is the foundation in designing secure schemes:

- parameter and key generation
- probabilistic encryption and signatures
- modeling of hash functions
- analyzing attacks and security
- protecting implementations

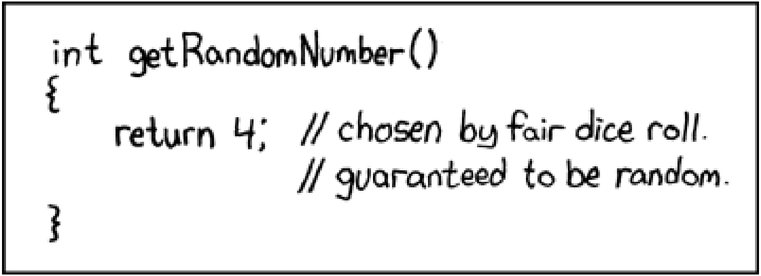
Entropy

Let $\mathcal{X}$ be an alphabet. For example, we have $\mathcal{X} = \{0, 1\}$ when flipping a coin and $\mathcal{X} = \{1, 2, 3, 4, 5, 6\}$ when rolling a die.

Let $p(x)$ be a probability distribution over $\mathcal{X}$ s.t. $p\colon \mathcal{X} \to [0, 1]$. We can e.g. assume that p is the uniform distribution over $\mathcal{X}$.

Let X be a random variable. The *(bit) entropy* H of X with respect to probability distribution p over alphabet $\mathcal{X}$ is defined as $H(X) = -\sum_{x\in\mathcal{X}} p(x)\log_2 p(x) = \mathbb{E}[-\log_2 p(X)]$.

Entropy


int getRandomNumber()
{
return 4; // chosen by fair dice roll.
// guaranteed to be random.
}

Entropy

Examples

- Let $\mathcal{X} = \{0, 1\}$ where $p(0) = 0, p(1) = 1$. Entropy is 0 bits.

Entropy

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- Let $\mathcal{X} = \{0, 1\}$ where $p(0) = p(1) = 1/2$. Entropy is 1 bit.

Entropy

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- Let $\mathcal{X} = \{0, 1\}$ where $p(0) = 1/3, p(1) = 2/3$. Entropy is $-(1/3 \cdot -1.584 + 2/3 \cdot -0.584) = 0.92$ bits.

Entropy

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- Let $\mathcal{X} = \{0, 1\}$ where $p(0) = 1/4, p(1) = 3/4$. Entropy is $-(1/4 \cdot -2 + 3/4 \cdot -0.42) = 0.81$ bits.

Entropy

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- Let $\mathcal{X} = \{0, 1\}$ where $p(0) = 1/4, p(1) = 3/4$. Entropy is $-(1/4 \cdot -2 + 3/4 \cdot -0.42) = 0.81$ bits.
- Let $\mathcal{X} = \{0, 1\}^\lambda$, uniform distribution. Entropy is λ bits.

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Security Parameter

We estimate the security of a scheme by how many bit-operations are needed to break it.

If an AES-128 key is sampled uniformly at random, then it takes 2^{128} trials to guess the right key.

We do not know of more efficient attacks against AES-128 than brute force guessing the key.

Security Parameter

We have more efficient algorithms for computing discrete logarithms. The generic algorithms run in time $\approx \sqrt{\mathbb{G}_p}$ ops.

Elliptic curves are generic groups without more structure. Need groups of prime size 256 bits to get 128 bits of security.

Security Parameter

We have even more efficient algorithms for computing discrete logarithms in finite fields and factoring bi-primes.

For finite field DH and DSA over $\mathbb{F}_p$ and for RSA encryption and signatures over $\mathbb{Z}_n$ we need p and n to be of 3072 bits.

Security Parameter

The computing power of the Bitcoin blockchain network is roughly 2^{60} operations per second. 2^{85} operations per year.

RSA-1024 has only 80 bits of security. We think this has been breakable by the NSA for at least ten years already.

We estimate that 2^{128} operations are infeasible even when using all computing power on Earth for the rest of the universe's lifetime. Quantum computers are not faster, but quantum algorithms might be more efficient.

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Sources of Randomness

To generate *real* randomness, we need a source of entropy:

- temperature measurements
- acoustic noise
- air turbulence
- electromagnetic radiation

These sources are hard to come by, measure, and analyse.

Random Numbers - Numberphile

Figure: `https://www.youtube.com/watch?v=SxP30euw3-0`

Sources of Randomness

What modern computers do today:

- keyboard timings
- mouse movements
- disk and network activity
- lava lamps*

It is recommended to use more than only one source. The operating system usually mixes several of the above.

Sources of Randomness

Figure: Cloudflare lava lamps: https://www.cloudflare.com/en-gb/learning/ssl/lava-lamp-encryption

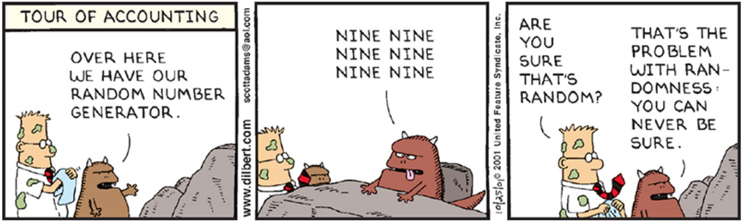
Bad Sources of Randomness

Sources that you should not use:

- the (exact) time of day (in μs))
- precomputed factory seed files
- process id or other environment variables
- whatever your “new friend” told you to use

If you extract too much randomness within a short time frame, then the entropy of fresh samples goes down.

Sources of Randomness



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Pseudorandom Number Generators

Pseudorandom Number Generators are deterministic algorithms that take as input a small sequence of *real* random bits and expand it into long sequences of *pseudorandom* bits streams.

A PRNG can perform three operations:

1. **init()** Initializes the internal state of the PRNG
2. **refresh(R)** Updates the state with randomness R
3. **next(N)** Returns N pseudorandom bits and refresh

Security Concerns

We want the following security properties of a PRNG:

1. *forward secrecy* means that previously generated pseudorandom bits are impossible to recover

2. *prediction resistance* means that future pseudorandom bits are impossible to predict

Under the Hood

Given a source of *real* randomness, the PRNGs we use today takes that as input and uses symmetric ciphers (e.g. AES) or hash-functions (e.g. SHA-2) to generate pseudorandom bits.

Non-Cryptographic PRNGs

Be aware that most programming languages provide non-cryptographic PRNGs by default. These PRNGs output *random-looking* numbers that might be predictable given e.g. a few samples or by running statistical tests on the output.

Some classic non-cryptographic PRNGs that people use:

- *Mersenne Twister* (Python, PHP, Ruby, Pascal,...)
- *Linear Congruential Generator* (Java, Python, Rust,...)
- *rand* and *drand48* (libc), *rand* and *mt_rand* (PHP)

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Schnorr Signatures

Let $\mathbb{G}$ be a group of prime order p and let g be a generator for $\mathbb{G}$. Denote by pp the public parameters $(\mathbb{G}, g, p)$.

Let H be a cryptographic hash function that outputs uniformly random elements in $\mathbb{Z}_p$.

Let the secret key $\mathsf{sk} \leftarrow\$ \mathbb{Z}_p$ be sampled uniformly at random, and let the public key be $\mathsf{pk} = g^{\mathsf{sk}}$, where pk is made public.

Schnorr Signatures

The Schnorr signature of message m is computed as:

1. Sample random $r \leftarrow\!\!\$ \mathbb{Z}_p$ and compute $R = g^r$.
2. Compute the output challenge as $c = H(\mathsf{pp}, \mathsf{pk}, m, R)$.
3. Compute the response $z = r - c \cdot \mathsf{sk}$. Output $\sigma = (c, z)$.

To verify the signature, compute $R' = g^z \cdot \mathsf{pk}^c$ and check if $c \stackrel{?}{=} H(\mathsf{pp}, \mathsf{pk}, m, R')$. If correct, accept, and otherwise reject.

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ElGamal Encryption

Let $\mathsf{pp} = (\mathbb{G}, g, p)$ as above. Sample uniform $\mathsf{sk} \leftarrow\!\!\$\ \mathbb{Z}_p$ and compute $\mathsf{pk} = g^{\mathsf{sk}}$, where pk is made public.

The ElGamal scheme, with $m \in \mathbb{G}$, works as follows:

Enc : Sample a random $x \leftarrow\!\!\$\ \mathbb{Z}_p$ and compute the ciphertext as $X = g^x$ and $Y = \mathsf{pk}^x \cdot m$.

Dec : Decrypt the ciphertext (X, Y) to get the message m as $m = Y \cdot X^{-\mathsf{sk}}$.

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RSA Cryptosystem

Sample large random prime numbers p and q and compute product $n = p \cdot q$. Compute $\phi(n) = (p-1) \cdot (q-1)$.

Choose integer e (co-prime with $\phi(n)$) and compute d such that $e \cdot d \equiv 1 \mod \phi(n)$. Let $\mathsf{sk} = (p, q, d)$ and $\mathsf{pk} = (n, e)$.

The RSA encryption scheme, with $m \in \mathbb{Z}_n$, works as follows:

Enc : Use (randomized) padding scheme μ to compute the ciphertext $c \equiv \mu(m)^e \mod n$.

Dec : Decrypt the ciphertext c to get the message m as the inverse padding $\mu^{-1}(c^d \mod n)$.

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Secure Hash Functions

When proving the security of cryptographic schemes that use hash functions H as underlying building blocks, we often model H as *random oracles* with an internal table of values.

You can read more about random oracles at Matthew Green's blog on cryptographic engineering:
https://blog.cryptographyengineering.com/2011/09/29/what-is-random-oracle-model-and-why-3

Questions?